

# BI-LIPSCHITZ $\mathcal{A}$ -EQUIVALENCE OF $\mathcal{K}$ -EQUIVALENT MAP-GERMS

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*Dedicated to Professor Heisuke Hironaka on the occasion of his 80th birthday*

**ABSTRACT.** In this paper, two sufficient conditions are provided for given two  $\mathcal{K}$ -equivalent map-germs to be bi-Lipschitz  $\mathcal{A}$ -equivalent. These are Lipschitz analogues of the known results on  $C^r$   $\mathcal{A}$ -equivalence ( $0 \leq r \leq \infty$ ) for given two  $\mathcal{K}$ -equivalent map-germs. As a corollary of one of our results, a Lipschitz version of the well-known Fukuda-Fukuda theorem is provided.

## 1. INTRODUCTION

The classification problem of generic singularities of  $C^\infty$  mappings up to  $C^\infty$  coordinate transformations is one of the most important problems in Singularity Theory. However, since it is impossible to obtain countably many list on this problem, the classification of singularities of  $C^\infty$  mappings by weaker equivalence relations are often studied. The  $C^r$   $\mathcal{A}$ -equivalence ( $0 \leq r < \infty$ ) has been well-studied. The bi-Lipschitz  $\mathcal{A}$ -equivalence is also one of the weaker equivalence relations and the classification problem with respect to bi-Lipschitz  $\mathcal{A}$ -equivalence is an interesting subject in the recent development of metric singularity theory (see for instance [2, 3, 4, 8]). In this paper we give sufficient conditions for the bi-Lipschitz  $\mathcal{A}$ -equivalence of  $\mathcal{K}$ -equivalent map-germs, providing Lipschitz versions of results with respect to the  $C^r$   $\mathcal{A}$ -equivalence given in [13, 15] where  $0 \leq r \leq \infty$ .

A map  $\varphi : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^p$  is said to be *Lipschitz* if there exists a constant  $c > 0$  such that the following holds, where  $U$  is an open set of  $\mathbb{R}^n$ .

$$\|\varphi(x) - \varphi(y)\| \leq c\|x - y\| \quad \text{for any } x, y \in U.$$

A Lipschitz map  $\varphi : U \subset \mathbb{R}^n \rightarrow \varphi(U) \subset \mathbb{R}^p$  is called a *bi-Lipschitz homeomorphism* if  $n = p$  and  $\varphi$  has a Lipschitz inverse. Two  $C^\infty$  map-germs  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  are said to be *bi-Lipschitz  $\mathcal{A}$ -equivalent* if there exist germs of bi-Lipschitz homeomorphisms  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and  $t : (\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^p, 0)$  such that  $f = t \circ g \circ s$ .

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Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map-germ. Any  $C^\infty$  map-germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  such that  $\Phi(x, 0) = f(x)$  is called a  $C^\infty$  deformation-germ of  $f$ .

**Definition 1.** Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map-germ. A  $C^\infty$  deformation-germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  is said to be *Lipschitz trivial* if there exist  $L$ -stratifications in the sense of [12, 16, 17],  $\mathcal{S}$  of  $\mathbb{R}^n \times \mathbb{R}^k$ ,  $\mathcal{T}$  of  $\mathbb{R}^p \times \mathbb{R}^k$  and  $\{\mathbb{R}^k\}$  of  $\mathbb{R}^k$  such that the following three conditions are satisfied:

- (i) The map-germ  $(\Phi, \pi) : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^k, (0, 0))$  is a stratified map with respect to  $\mathcal{S}$  and  $\mathcal{T}$ , where  $\pi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^k, 0)$  is the canonical projection.
- (ii) The canonical projection  $\pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^k, 0)$  is a stratified map with respect to  $\mathcal{T}$  and  $\{\mathbb{R}^k\}$ .
- (iii) There exist germs of stratified Lipschitz vector fields  $\xi_i + \frac{\partial}{\partial \lambda_i}$  at the origin in  $\mathbb{R}^n \times \mathbb{R}^k$  with respect to  $\mathcal{S}$  and  $\eta_i + \frac{\partial}{\partial \lambda_i}$  at the origin in  $\mathbb{R}^p \times \mathbb{R}^k$  with respect to  $\mathcal{T}$  such that

$$\xi_i + \frac{\partial}{\partial \lambda_i} \text{ lifts } \eta_i + \frac{\partial}{\partial \lambda_i} \text{ with respect to } (\Phi, \pi),$$

where  $\frac{\partial}{\partial \lambda_i}$  is the trivial vector field of  $\mathbb{R}^k$  with respect to the  $i$ -th coordinate function  $\lambda_i$  of the standard coordinate neighborhood  $(\mathbb{R}^k, (\lambda_1, \dots, \lambda_k))$ .

Definition 1 was motivated by the definition of *Thom trivial* deformation given by Nishimura in [13]. A deformation is Thom trivial if there exists a  $C$ -regular stratification in the sense of K. Bekka ([1]), satisfying conditions (i) and (ii) above where  $(\Phi, \pi)$  is a Thom map. For any Thom trivial deformation we can apply Thom's second isotopy lemma ([1, 6, 11, 18]) which implies that  $(\Phi, \pi)$  is topologically equivalent to  $(f, \pi)$ . To obtain a similar condition to that for the Lipschitz case, we replace  $C$ -regular stratifications by  $L$ -stratifications (in the sense of Mostowski [12]) and we add the condition (iii) above, because there seems to have been no bi-Lipschitz versions of Thom's second isotopy lemma.

It follows from Definition 1 that for any Lipschitz trivial deformation-germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$ , there exist germs of bi-Lipschitz homeomorphisms  $h : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^k, (0, 0))$  and  $H : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^k, (0, 0))$  such that the following diagram is commutative:

$$(1) \quad \begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) & \xrightarrow{(\Phi, \pi)} & (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) & \xrightarrow{\pi'} & (\mathbb{R}^k, 0) \\ h \downarrow & & H \downarrow & & \parallel \\ (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) & \xrightarrow{(f, \pi)} & (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) & \xrightarrow{\pi'} & (\mathbb{R}^k, 0), \end{array}$$

where  $\pi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^k, 0)$  and  $\pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^k, 0)$  are canonical projections. The main result of this paper is the following:

**Theorem 1.** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs. Suppose that the following three conditions are satisfied.

- (i) There exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ .
- (ii) The  $C^\infty$  map-germ  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

- is a Lipschitz trivial deformation-germ of  $f$ .  
 (iii) The  $C^\infty$  map-germ  $G : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a Lipschitz trivial deformation-germ of  $g$ .

Then,  $f$  and  $g$  are bi-Lipschitz  $\mathcal{A}$ -equivalent.

The condition (i) in Theorem 1 is equivalent to say that  $f$  and  $g$  are  $\mathcal{K}$ -equivalent (see [9]).

**Definition 2.** A  $C^\infty$  map-germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  is said to be *Lipschitz stable* if any  $C^\infty$  deformation-germ of  $f$  is Lipschitz trivial.

As a corollary of Theorem 1, the following Lipschitz version of Fukuda-Fukuda theorem [5] is obtained.

**Corollary 1.** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two Lipschitz stable map-germs. Suppose that there exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ . Then,  $f$  and  $g$  are bi-Lipschitz  $\mathcal{A}$ -equivalent.

In the case that both the given two map-germs  $f$  and  $g$  are of rank zero, Theorem 1 can be strengthened as follows.

**Theorem 2.** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs of rank zero. Suppose that the following two conditions are satisfied.

- (i) There exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ .
- (ii) The  $C^\infty$  map-germ  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a Lipschitz trivial deformation-germ of  $f$ .

Then,  $f$  and  $g$  are bi-Lipschitz  $\mathcal{A}$ -equivalent.

**Remark 1.** Let  $f, g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  given by

$$f(x, y) = (x, y^3 + xy) \quad \text{and} \quad g(x, y) = (x, y^3).$$

It is easily seen that  $f(x, y) = M(x, y)g(x, y)$  with  $M : (\mathbb{R}^2, 0) \rightarrow (GL(2, \mathbb{R}), E_2)$ . Moreover, it is well known that  $f$  is infinitesimally stable, hence any  $C^\infty$  deformation germ of  $f$  is  $C^\infty$  trivial (and hence Lipschitz trivial). However,  $f$  and  $g$  are not bi-Lipschitz  $\mathcal{A}$ -equivalent. This example shows that rank zero hypothesis is essential in Theorem 2.

In Section 2,  $C^r$  versions of Theorem 1, Corollary 1 and Theorem 2 are reviewed where  $0 \leq r \leq \infty$ . In Section 3, a strategy to show Theorems 1 and 2 is explained. Theorems 1 and 2 are proved in Section 4.

2. THE  $C^r$  VERSIONS ( $0 \leq r \leq \infty$ )

In this section, for the readers' convenience, we review  $C^r$  versions of Theorem 1, Corollary 1 and Theorem 2 ( $0 \leq r \leq \infty$ ) briefly. For details on  $C^r$  versions ( $1 \leq r \leq \infty$ ), refer to [15]; and on  $C^0$  versions, refer to [13] (see also [14] where [13, 15] and related results are summarized).

- Definition 3.** (i) Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs.
- (a) The given  $f$  and  $g$  are said to be  $C^r$   $\mathcal{A}$ -equivalent ( $1 \leq r \leq \infty$ ) if there exist germs of  $C^r$  diffeomorphisms  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and  $t : (\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^p, 0)$  such that  $f = t \circ g \circ s$ .
  - (b) The given  $f$  and  $g$  are said to be *topologically  $\mathcal{A}$ -equivalent* if there exist germs of homeomorphisms  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and  $t : (\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^p, 0)$  such that  $f = t \circ g \circ s$ .
- (ii) Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map-germ. A  $C^\infty$  deformation-germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  is said to be  $C^r$   $\mathcal{A}$ -trivial ( $1 \leq r \leq \infty$ ) if there exist germs of  $C^r$  diffeomorphisms  $h : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^k, (0, 0))$  and  $H : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^k, (0, 0))$  such that the diagram (1) in Section 1 is commutative.

2.1. The  $C^r$  versions ( $1 \leq r \leq \infty$ ).

**Theorem 3** ([15]). Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs. Suppose that the following three conditions are satisfied.

- (i) There exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ .
- (ii) The  $C^\infty$  map-germ  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a  $C^r$   $\mathcal{A}$ -trivial deformation-germ of  $f$ .

- (iii) The  $C^\infty$  map-germ  $G : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a  $C^r$   $\mathcal{A}$ -trivial deformation-germ of  $g$ .

Then,  $f$  and  $g$  are  $C^r$   $\mathcal{A}$ -equivalent ( $1 \leq r \leq \infty$ ).

A  $C^\infty$  map-germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  is said to be  $C^r$  *stable* ( $1 \leq r \leq \infty$ ) if any  $C^\infty$  deformation of  $f$  is  $C^r$   $\mathcal{A}$ -trivial ( $1 \leq r \leq \infty$ ). It is common in the literature to call a  $C^\infty$  stable map-germ just by stable map-germ.

**Corollary 2** ([15]). Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^r$  stable map-germs ( $1 \leq r \leq \infty$ ). Suppose that there exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ . Then,  $f$  and  $g$  are  $C^r$   $\mathcal{A}$ -equivalent.

Notice that for  $r = \infty$ , Corollary 2 is the classical Mather's classification theorem for stable map-germs (see [10]).

**Theorem 4** ([15]). Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs of rank zero. Suppose that the following two conditions are satisfied.

- (i) There exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ .

- (ii) The  $C^\infty$  map-germ  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a  $C^r$   $\mathcal{A}$ -trivial deformation-germ of  $f$ .

Then,  $f$  and  $g$  are  $C^r$   $\mathcal{A}$ -equivalent ( $1 \leq r \leq \infty$ ).

## 2.2. The $C^0$ version.

**Definition 4.** Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map-germ. A  $C^\infty$  deformation-germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  is said to be *Thom trivial* if there exist  $C$ -regular stratifications in the sense of [1],  $\mathcal{S}$  of  $\mathbb{R}^n \times \mathbb{R}^k$ ,  $\mathcal{T}$  of  $\mathbb{R}^p \times \mathbb{R}^k$  and  $\{\mathbb{R}^k\}$  of  $\mathbb{R}^k$  such that the following two conditions are satisfied:

- (i) The map-germ  $(\Phi, \pi) : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^k, (0, 0))$  is a Thom map-germ with respect to  $\mathcal{S}$  and  $\mathcal{T}$ , where  $\pi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^k, 0)$  is the canonical projection.
- (ii) The canonical projection  $\pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^k, 0)$  is a stratified map-germ with respect to  $\mathcal{T}$  and  $\{\mathbb{R}^k\}$ .

It is known that there exists a controlled tube system for any  $C$ -regular stratification (see [1]). By Thom's second isotopy lemma, it follows that for any Thom trivial deformation-germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$ , there exist germs of homeomorphisms  $h : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^k, (0, 0))$  and  $H : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^k, (0, 0))$  such that the diagram (1) in Section 1 is commutative.

**Theorem 5** ([13]). *Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs. Suppose that the following three conditions are satisfied.*

- (i) *There exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ .*
- (ii) *The  $C^\infty$  map-germ  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by*

$$F(x, \lambda) = f(x) - M(x)\lambda$$

*is a Thom trivial deformation-germ of  $f$ .*

- (iii) *The  $C^\infty$  map-germ  $G : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by*

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

*is a Thom trivial deformation-germ of  $g$ .*

Then,  $f$  and  $g$  are topologically  $\mathcal{A}$ -equivalent.

A  $C^\infty$  map-germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  is said to be *MT-stable* if the jet extension of it is multitransverse to the Thom-Mather canonical stratification of the jet space (for the Thom-Mather canonical stratification, see for instance [6, 11, 18]). By definition, it follows that any  $C^\infty$  deformation of an MT-stable map-germ is Thom trivial. Hence, the following Fukuda-Fukuda theorem follows from Theorem 5.

**Corollary 3** (Fukuda-Fukuda theorem [5]). *Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be MT-stable map-germs. Suppose that there exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ . Then,  $f$  and  $g$  are topologically  $\mathcal{A}$ -equivalent.*

**Theorem 6** ([13]). *Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs of rank zero. Suppose that the following two conditions are satisfied.*

- (i) *There exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ .*
- (ii) *The  $C^\infty$  map-germ  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  given by*

$$F(x, \lambda) = f(x) - M(x)\lambda$$

*is a Thom trivial deformation-germ of  $f$ .*

*Then,  $f$  and  $g$  are topologically  $\mathcal{A}$ -equivalent.*

### 3. STRATEGY

The strategy to prove Theorems 1 and 2 is almost the same as the strategies given in [13, 15]. In [13] (resp., [15]), the strategy is stated in terms of germs of homeomorphisms (resp., germs of  $C^r$  diffeomorphisms ( $1 \leq r \leq \infty$ )). It is possible to be stated also in terms of germs of bi-Lipschitz homeomorphisms as shown below.

Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map-germs. Suppose that there exist a germ of  $C^\infty$  diffeomorphism  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map-germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)g(s(x))$ . Then, we let  $F : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p, 0)$  be the map-germ defined by

$$F(x, \lambda) = f(x) - M(x)\lambda,$$

where  $\mathbb{R}_\lambda^p$  (resp.,  $\mathbb{R}_y^p$ ) is the  $p$ -dimensional Euclidean space  $\mathbb{R}^p$  which plays the role of the parameter space (resp., the target space) of  $F$ . Suppose furthermore that  $F$  is Lipschitz trivial. Then, it follows from Definition 1 that there exist  $L$ -stratifications in the sense of [12, 16, 17],  $\mathcal{S}$  of  $\mathbb{R}^n \times \mathbb{R}_\lambda^p$ ,  $\mathcal{T}$  of  $\mathbb{R}_y^p \times \mathbb{R}_\lambda^p$  and  $\{\mathbb{R}_\lambda^p\}$  of  $\mathbb{R}_\lambda^p$  such that the three conditions in Definition 1 are satisfied. By the condition (iii) of Definition 1, there exist germs of bi-Lipschitz homeomorphisms  $h : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0))$  and  $H : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0))$  such that the diagram (1) in Section 1 is commutative. By the commutative diagram (1) we may put

$$h(x, \lambda) = (h_1(x, \lambda), \lambda) \quad \text{and} \quad H(y, \lambda) = (H_1(y, \lambda), \lambda).$$

**Lemma 3.1** ([15], Lemma 2.1).  *$f(h_1(x, g(s(x)))) = H_1(0, g(s(x)))$ .*

**Lemma 3.2** ([13], Lemma 2.2). *If the map germ from  $(\mathbb{R}_\lambda^p, 0)$  to  $(\mathbb{R}_y^p, 0)$  defined by*

$$(2) \quad \lambda \mapsto H_1(0, \lambda)$$

*is a germ of homeomorphism, then the map germ defined by*

$$(3) \quad (x, \lambda) \mapsto (h_1(x, \lambda), H_1(0, \lambda))$$

*maps the germ of the set  $(F^{-1}(0), (0, 0))$  onto the germ of the graph of  $f$  at  $(0, 0)$ .*

**Lemma 3.3.** *If the map germ (2) is bi-Lipschitz, then the endomorphism germ of  $(\mathbb{R}^n, 0)$  defined by*

$$(4) \quad x \mapsto h_1(x, g(s(x)))$$

*is also bi-Lipschitz.*

*Proof of Lemma 3.3.* The proof is almost the same as the proof of Lemma 2.3 in [13]. The map-germ (4) can be decomposed as follows.

$$(5) \quad x \mapsto (x, g(s(x))) \mapsto (h_1(x, g(s(x))), H_1(0, g(s(x)))) \mapsto h_1(x, g(s(x))).$$

If the map-germ (2) is bi-Lipschitz, then the map-germ (3) is also bi-Lipschitz. Since the first map-germ of (5) is the germ of the graph of  $g \circ s$  at  $(0, 0)$ , the composition of the first and the second map-germ in (5) is also a germ of bi-Lipschitz homeomorphism to the germ of the set

$$(6) \quad (\{(h_1(x, g(s(x))), H_1(0, g(s(x)))) \mid x \in \mathbb{R}^n\}, (0, 0))$$

with respect to the induced metric from  $\mathbb{R}^n \times \mathbb{R}_y^p$ . The last map-germ in (5) is the restriction of the canonical projection  $(\mathbb{R}^n \times \mathbb{R}_y^p, (0, 0)) \rightarrow (\mathbb{R}^n, 0)$  to (6). By Lemma 3.2, (6) is equal to the germ of the set  $(\text{graph}(f), (0, 0))$ . Hence, (4) is bi-Lipschitz.  $\square$

**Lemma 3.4.** *Let  $T_0$  be the stratum of  $\mathcal{T}$  containing the origin  $(0, 0)$ . If  $T_0$  is transverse to the linear space  $\{0\} \times \mathbb{R}_\lambda^p$ , then the map-germ (2) is bi-Lipschitz.*

*Proof of Lemma 3.4.* It is clearly seen that the map-germ (2) is Lipschitz even if  $T_0$  is not transverse to  $\{0\} \times \mathbb{R}_\lambda^p$ .

We show that if  $T_0$  is transverse to  $\{0\} \times \mathbb{R}_\lambda^p$ , then the map-germ (2) has its inverse which is also Lipschitz. It is easily seen that the map-germ (2) has its inverse if  $T_0$  is transverse to  $\{0\} \times \mathbb{R}_\lambda^p$ . For each  $i$  ( $1 \leq i \leq p$ ), let  $\eta_i + \frac{\partial}{\partial \lambda_i}$  be the germ of stratified Lipschitz vector field such that the condition (iii) of Definition 1 is satisfied. Moreover, let  $\Theta_i : (\mathbb{R} \times \mathbb{R}_y^p, (0, 0)) \rightarrow (\mathbb{R}_y^p, 0)$  be the germ of local flow for  $\eta_i$ . Then, the following holds where  $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}_\lambda^p$ .

$$\Theta_1(\lambda_1; \Theta_2(\lambda_2; \dots, \Theta_p(\lambda_p, H_1(0, \lambda)) \dots) = 0.$$

For any sufficiently small  $\lambda = (\lambda_1, \dots, \lambda_p)$  in  $\mathbb{R}_\lambda^p$  and  $y \in \mathbb{R}_y^p$ , set  $\Theta(\lambda, y) = \Theta_1(\lambda_1; \Theta_2(\lambda_2; \dots, \Theta_p(\lambda_p, y) \dots)$ . Let  $\lambda = (\lambda_1, \dots, \lambda_p), \tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_p)$  be sufficiently small two points of  $\mathbb{R}_\lambda^p$ . Then, since  $\eta_i$  is Lipschitz for any  $i$  ( $1 \leq i \leq p$ ), there exists a positive constant  $L$  such that the following holds for any  $j$  ( $1 \leq j \leq p$ ), where  $y_{j,\lambda}$  (resp.,  $y_{j,\tilde{\lambda}}$ ) stands for  $\Theta((0, \dots, 0, \lambda_j, \dots, \lambda_p), H_1(0, \lambda))$  (resp.,  $\Theta((0, \dots, 0, \lambda_j, \dots, \lambda_p), H_1(0, \tilde{\lambda}))$ ) and  $y_{p+1,\lambda}$  (resp.,  $y_{p+1,\tilde{\lambda}}$ ) stands for  $H_1(0, \lambda)$  (resp.,  $H_1(0, \tilde{\lambda}))$ .

$$\begin{aligned} \|y_{j,\lambda} - y_{j,\tilde{\lambda}}\| &= \|\Theta_j(\lambda_j, y_{j+1,\lambda}) - \Theta_j(\lambda_j, y_{j+1,\tilde{\lambda}})\| \\ &\leq \|y_{j+1,\lambda} - y_{j+1,\tilde{\lambda}}\| + L \left| \int_0^{\lambda_j} \left\| \Theta_j(s, y_{j+1,\lambda}) - \Theta_j(s, y_{j+1,\tilde{\lambda}}) \right\| ds \right|. \end{aligned}$$

Since  $\lambda = (\lambda_1, \dots, \lambda_p)$  is sufficiently small, we may assume that  $|\lambda_j| < 1$  for any  $j$  ( $1 \leq j \leq p$ ). Thus, by Gronwall's inequality (for instance, see [7]), the following holds:

$$\begin{aligned} \|\Theta(\lambda, H_1(0, \tilde{\lambda}))\| &= \|\Theta(\lambda, H_1(0, \lambda)) - \Theta(\lambda, H_1(0, \tilde{\lambda}))\| \\ &= \|y_{1,\lambda} - y_{1,\tilde{\lambda}}\| \\ &\leq e \|y_{2,\lambda} - y_{2,\tilde{\lambda}}\| \\ &\vdots \\ &\leq e^p \|y_{p+1,\lambda} - y_{p+1,\tilde{\lambda}}\| \\ &= e^p \|H_1(0, \lambda) - H_1(0, \tilde{\lambda})\|. \end{aligned}$$

For any  $j$  ( $1 \leq j \leq p$ ), set  $\eta_j(y, \lambda) = \sum_{i=1}^p \eta_{ij}(y, \lambda) \frac{\partial}{\partial y_i}$ , where  $\frac{\partial}{\partial y_i}$  is the trivial vector field of  $\mathbb{R}_y^p$  with respect to the  $i$ -th coordinate function  $y_i$  of the standard coordinate neighborhood  $(\mathbb{R}_y^p, (y_1, \dots, y_p))$ . Let the following  $p$  by  $p$  matrix be denoted by  $M(y, \lambda)$ :

$$M(y, \lambda) = \begin{pmatrix} \eta_{11}(y, \lambda) & \cdots & \eta_{1p}(y, \lambda) \\ \vdots & \ddots & \vdots \\ \eta_{p1}(y, \lambda) & \cdots & \eta_{pp}(y, \lambda) \end{pmatrix}.$$

Since  $T_0$  is transverse to  $\{0\} \times \mathbb{R}_\lambda^p$ , we have that  $\eta_1(0, 0), \dots, \eta_p(0, 0)$  are linearly independent. This implies that  $\min_{\|\tilde{\lambda}\|=1} \|M(0, 0)\tilde{\lambda}\|$  is positive. By the continuity of  $\eta_i$  with respect to  $y, \lambda \in \mathbb{R}^p$ , it follows that there exists a positive constant  $M$  such that

$$(7) \quad 0 < M \leq \min_{\|\tilde{\lambda}\|=1} \|M(y, \lambda)\tilde{\lambda}\|,$$

for any sufficiently small  $y, \lambda \in \mathbb{R}^p$ .

Let  $\varepsilon > 0$  be a sufficiently small real number and let  $\psi = ({}_1\psi, {}_2\psi) : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}_\lambda^p \times \mathbb{R}_\lambda^p$  be a regular curve such that  $\psi(0) \in \Delta$  and  ${}_1\psi'(0) \neq {}_2\psi'(0)$ , where  $\Delta$  is the diagonal set of  $\mathbb{R}_\lambda^p \times \mathbb{R}_\lambda^p$ . Set  $y_j \circ {}_i\psi = {}_i\psi_j$  ( $1 \leq i \leq 2, 1 \leq j \leq p$ ). Then, the following equality holds:

$$\Theta_1({}_2\psi_1(t) - {}_1\psi_1(t); \dots, \Theta_p({}_2\psi_p(t) - {}_1\psi_p(t); \Theta({}_1\psi(t), H_1(0, {}_2\psi(t)))) \cdots) = 0.$$

By differentiating both sides of the above equality with respect to  $t$  at  $t = 0$ , we have

$$M(0, {}_1\psi(0))({}_2\psi'(0) - {}_1\psi'(0)) + \lim_{t \rightarrow 0} \frac{\Theta({}_1\psi(t), H_1(0, {}_2\psi(t)))}{t} = 0.$$

Combining this equality and the Maclaurin expansion of  ${}_2\psi(t) - {}_1\psi(t)$ , we have the following:

$$(8) \quad \lim_{t \rightarrow 0} \frac{\|\Theta({}_1\psi(t), H_1(0, {}_2\psi(t)))\|}{\|{}_2\psi(t) - {}_1\psi(t)\|} = \left\| M(0, {}_1\psi(0)) \frac{{}_2\psi'(0) - {}_1\psi'(0)}{\|{}_2\psi'(0) - {}_1\psi'(0)\|} \right\|.$$

Let  $\Psi : \mathbb{R}_\lambda^p \times \mathbb{R}_\lambda^p - \Delta \rightarrow \mathbb{R}$  be the function defined by

$$\Psi(\lambda, \tilde{\lambda}) = \frac{\|\Theta(\lambda, H_1(0, \tilde{\lambda}))\|}{\|\tilde{\lambda} - \lambda\|}.$$

Moreover, let  $\tilde{\pi} : B \rightarrow \mathbb{R}_\lambda^p \times \mathbb{R}_\lambda^p$  be the blowup centered at  $\Delta$ . The equality (8) shows that there exists the unique continuous function  $\tilde{\Psi} : B \rightarrow \mathbb{R}$  such that  $\tilde{\Psi} = \Psi \circ \tilde{\pi}$  on  $\mathbb{R}_\lambda^p \times \mathbb{R}_\lambda^p - \Delta$ . Thus, by the inequality (7), for any sufficiently small  $\lambda, \tilde{\lambda} \in \mathbb{R}_\lambda^p$  such that  $\lambda \neq \tilde{\lambda}$ , the following holds:

$$\frac{M}{2} \leq \frac{\|\Theta(\lambda, H_1(0, \tilde{\lambda}))\|}{\|\tilde{\lambda} - \lambda\|}.$$

Therefore, for any sufficiently small  $\lambda, \tilde{\lambda} \in \mathbb{R}_\lambda^p$ , we have the following inequality which shows that the inverse map-germ of (2) is actually Lipschitz.

$$\|\lambda - \tilde{\lambda}\| \leq e^p \frac{2}{M} \|H_1(0, \lambda) - H_1(0, \tilde{\lambda})\|.$$

□



## 4. PROOFS OF THEOREM 1 AND THEOREM 2

The proof of Theorem 1 (resp., Theorem 2) is almost the same as the proof of Theorem 1.2 (resp., Theorem 1.3) in [13]. Theorem 1.3 in [13] is proved by using terms of Thom trivial deformations,  $C$ -regular stratifications, Lemma 3.1 and the following two lemmas.

**Lemma 4.1** ([13], Lemma 2.3). *If the map germ from  $(\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0)$  defined by*

$$\lambda \mapsto H_1(0, \lambda)$$

*is a germ of homeomorphism, then the endomorphism germ of  $(\mathbb{R}^n, 0)$  defined by*

$$(9) \quad x \mapsto h_1(x, g(s(x)))$$

*is also a germ of homeomorphism.*

**Lemma 4.2.** *Suppose that  $F$  is Thom trivial. Let  $\mathcal{S}$ ,  $\mathcal{T}$  and  $\mathbb{R}_\lambda^p$  be the  $C$ -regular stratifications of  $\mathbb{R}^n \times \mathbb{R}_\lambda^p$ ,  $\mathbb{R}^p \times \mathbb{R}_\lambda^p$  and  $\{\mathbb{R}_\lambda^p\}$  respectively whose existence are guaranteed by Thom triviality of  $F$ . Let  $T_0$  be the stratum of  $\mathcal{T}$  containing the origin  $(0, 0)$ . If  $T_0$  is transverse to the linear space  $\{0\} \times \mathbb{R}_\lambda^p$ , then the map-germ  $\lambda \mapsto H_1(0, \lambda)$  is a germ of homeomorphism.*

For the proof of Theorem 1.2 in [13], since we need to consider perturbations of  $F(x, \lambda) = f(x) - M(x)\lambda$  and  $g(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$  by matrices, two modifications of Lemma 4.2 are used. By the proof of Lemma 3.4, it is not hard to prove that bi-Lipschitz version of these two modifications of Lemma 4.2 follows.

Therefore, by replacing “Thom trivial”, “ $C$ -regular stratifications” and Lemmas 4.1 and 4.2 (resp., Lemma 4.1 and two modifications of Lemma 4.2) with “Lipschitz trivial”, “ $L$ -regular stratifications” and Lemmas 3.3 and 3.4 (resp., Lemma 3.3 and the bi-Lipschitz version of two modifications of Lemma 4.2) respectively, the Proof of Theorem 1.3 (resp., Theorem 1.2) in [13] works well as the proof of Theorem 2 (resp., Theorem 1).  $\square$

**Remark 2.** To give sufficient conditions for bi-Lipschitz  $\mathcal{A}$ -equivalence of Lipschitz stable map-germs is a key step in the attempt to answer the following difficult open question in the subject of density of stable mappings:

**Question:** Is the set of Lipschitz stable mappings dense in  $C^\infty(M^n, N^p)$ , with the Whitney  $C^\infty$  topology, when  $(n, p)$  are outside the *nice dimensions*?

It is well known that  $C^0$  stable maps are dense in general and that  $C^1$  stability fails to be dense in the function space exactly when  $C^\infty$  stability fails to be dense, that is, outside the nice dimensions. However, few results are known about Lipschitz stable maps. Actually we know of no example of a Lipschitz stable map (or germ) which is not  $C^\infty$  stable.

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